Math 2050, highlight of Week 1

1. Motivating Question:

What is the system of real numbers $\mathbb{R}$?

1.1. **Algebraic properties of Real numbers.** We start with the algebraic properties of $(\mathbb{R}, +, \cdot)$:

- (a1) $\forall a, b \in \mathbb{R}$, we have $a + b = b + a$;
- (a2) $\forall a, b, c \in \mathbb{R}$, we have $(a + b) + c = a + (b + c)$;
- (a3) $\exists 0 \in \mathbb{R}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}$;
- (a4) $\forall a \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $a + b = b + a = 0$.
- (m1) $\forall a, b \in \mathbb{R}$, we have $a \cdot b = b \cdot a$;
- (m2) $\forall a, b, c \in \mathbb{R}$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- (m3) $\exists 1 \neq 0 \in \mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{R}$;
- (m4) $\forall a \neq 0 \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $a \cdot b = b \cdot a = 1$.
- (d) For all $a, b, c \in \mathbb{R}$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

Theorem 1.1. *(uniqueness) From the algebraic properties of $\mathbb{R}$, we have the following uniqueness of elements:*

- (i) *If $a, b \in \mathbb{R}$ are elements such that $a + b = a$, then $b = 0$.*
- (ii) *If $a, b \in \mathbb{R}$ are elements such that $a \neq 0$ and $a \cdot b = a$, then $b = 1$.*
- (iii) *Given $a \in \mathbb{R}$. If $b, c \in \mathbb{R}$ are such that $a + b = 1 = a + c$, then $b = c$.*
- (iv) *Given $0 \neq a \in \mathbb{R}$. If $b, c \in \mathbb{R}$ are such that $a \cdot b = 1 = a \cdot c$, then $b = c$.*

Remark 1.1. The importance of this Theorem is that the "zero" and "identity" elements are unique. Moreover, the additive and multiplicative inverse are unique. And hence, we may use $-a$ and a^{-1} to denote the inverse respectively.

With the inverse defined, we may proceed to define the "negative" operation. Namely, the subtraction:

$$a - b = a + (-b), \quad \forall a, b \in \mathbb{R};$$

and division:

$$a/b = a \cdot (b^{-1}), \quad \forall a, b \in \mathbb{R}, \; b \neq 0.$$

1.2. **Ordering properties of Real numbers.** Next, we would like to define a ordering properties of the real number which enables us to compare elements as well as define the inequalities, distance, etc. To do this, we let $\mathbb{P}$ be a subset of $\mathbb{R}$ such that the following holds:

(i) If $a \in \mathbb{R}$, then either $a = 0$, $a \in \mathbb{P}$ or $-a \in \mathbb{P}$;
(ii) if $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$;
(iii) if $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.

We call the set $\mathbb{P}$ to be set of positive numbers. With this definition, all simple inequality will hold (Check!).

With this in hand, we also define distance between $x, y \in \mathbb{R}$ by $|x-y|$ where $|a|$ of a real number $a \in \mathbb{R}$ is given by

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a < 0. \end{cases} \tag{1.1}$$

1.3. **Distinction between $\mathbb{R}$ and $\mathbb{Q}$.** In view of algebraic properties, we can see the necessity of improving the number systems:

- Natural number $\mathbb{N}$: Fail to obey addition rule;
- Integers $\mathbb{Z}$: Fail to obey multiplicative rule;
- Rational number $\mathbb{Q}$: Satisfies all rule!

Problem raised:

Theorem 1.2. *There is no $x \in \mathbb{Q}$ such that $x^2 = 2$.*

1.4. **Completeness of $\mathbb{R}$.**

Definition 1.1. *Let $A \subset \mathbb{R}$ be a subset, we say that A is bounded from above if there is $M \in \mathbb{R}$ such that for all $a \in A$, $a \leq M$.*

Analogously, we can define the notion of "bounded below" and "bounded". Clearly, there are no unique upper bound for a bounded set. We therefore look for the "best" one.

Definition 1.2. *Given a non-empty subset $S \subset \mathbb{R}$ which is bounded from above. A real number $u = \sup S$ (the least upper bound) if*

(1) *u is an upper bound of S;*
(2) *If v is another upper bound of S, then $v \geq u$.*

Remark: by (2), $\sup S$ is unique if exists.

The greatest lower bound ($\inf S$) is defined analogously for non-empty subset S which is bounded from below.

The completeness of $\mathbb{R}$:

For any non-empty subset S which is bounded from above, $\sup S$ *exists.*

Corollary 1.1. *For any non-empty subset S which is bounded from below,* $\inf S$ *exists.*

Important applications of completeness:

Theorem 1.3 (Archimedean properties)**.** *The set of natural number $\mathbb{N}$ is unbounded.*

Another crucial consequence to our motivating question (!!!):

Theorem 1.4. *There is $u \in \mathbb{R}$ such that $u^2 = 2$.*

Proof. Let $A = \{a \in \mathbb{R} : a^2 < 2\}$. The set A is non-empty since $0^2 = 0 < 2$ and hence $0 \in A$. Moreover, A is bounded from above by 2 since otherwise, there is $a > 2$ such that $4 < a^2 < 2$ which is impossible.

Therefore, completeness implies that $u = \sup A$ exists in $\mathbb{R}$. Since $1^2 = 1 < 2$, we have $u \geq 1 \in A$. We claim that $u^2 = 2$. Suppose on the contrary, we either have $u^2 < 2$ or $u^2 > 2$.

Case 1. $u^2 > 2$. We choose $\varepsilon > 0$ to be

$$\varepsilon = \min\left\{\frac{u^2-2}{2u}, u\right\} > 0.$$

By properties of $\sup A$, there is $a \in A$ such that $0 < u - \varepsilon < a$ and hence

$$(u-\varepsilon)^2 < a^2 < 2.$$

But the number $v = u - \varepsilon$ satisfies

$$v^2 = u^2 - 2\varepsilon u + \varepsilon^2 > u^2 - 2\varepsilon u > 2$$

which is impossible.

Case 2. $u^2 < 2$. We choose $\varepsilon > 0$ to be

$$\varepsilon = \min\left\{1, \frac{2-u^2}{2(2u+1)}\right\} > 0$$

Then the number $v = u + \varepsilon$ satisfies

$$\begin{aligned} v^2 &= u^2 + 2\varepsilon u + \varepsilon^2 \\ &\leq u^2 + \varepsilon(2u+1) \\ &\leq u^2 + \frac{2-u^2}{2} \\ &< 2. \end{aligned}$$

Therefore, $v \in A$ and hence $u + \varepsilon \leq u$ which is impossible. $\square$